Attenuation and Dispersion in Fibres

HTE - 15.03.2013

1. General

The schematic diagram of a fibre link and its various components are shown in Fig 1.1.

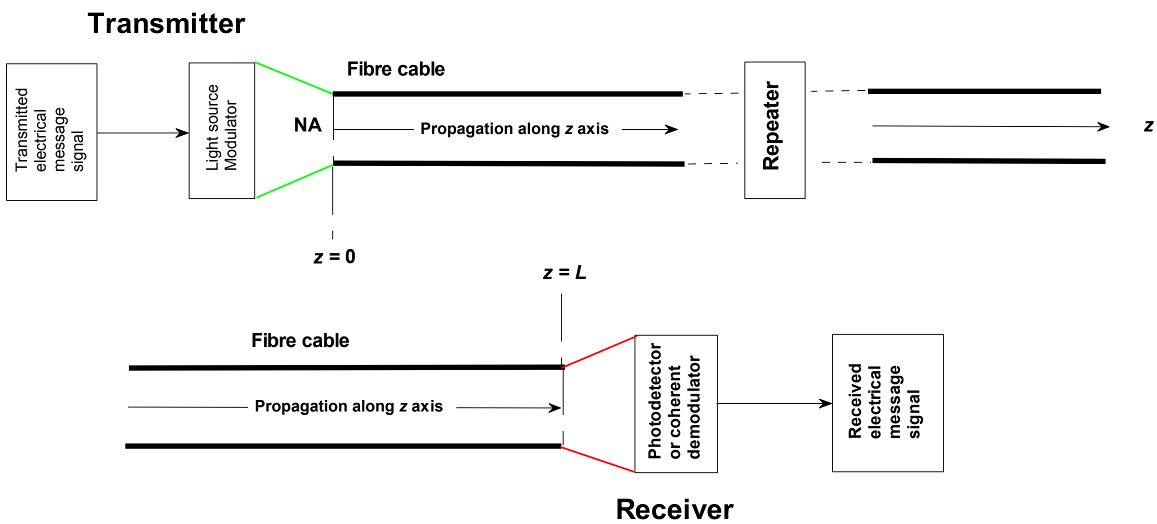

Fig. 1.1 Schematic diagram of a fibre link and its components.

Our interpretation of a fibre link is that, on the transmitter side, an electrical message signal modulates in some manner a light source of some type, thus the optical wavelength of the light source cats as a carrier. This modulated light is then launched into the fibre through the acceptance cone (NA) of the fibre. The fibre cable is usually quite long, so that there may be repeaters on the way, before reaching the receiver side. Finally on the receiver side, we employ photo detector ar another light source to demodulate the original electrical message signal. Our performance measures are, for a given SNR or probability of error, longer link lengths (L) (basically without repeaters) and the higher bit rates of the electrical message signal we can achieve over this link are regarded as better performance measures.

Since we are going to use the fibre cable in outdoor installations as well, some mechanical protection such as coating and strength elements will be required in order to make the fibre durable to these conditions. A typical mechanical structure is shown below in Fig. 1.2 (copied directly from Fig. 2.4 of Ref. [2])

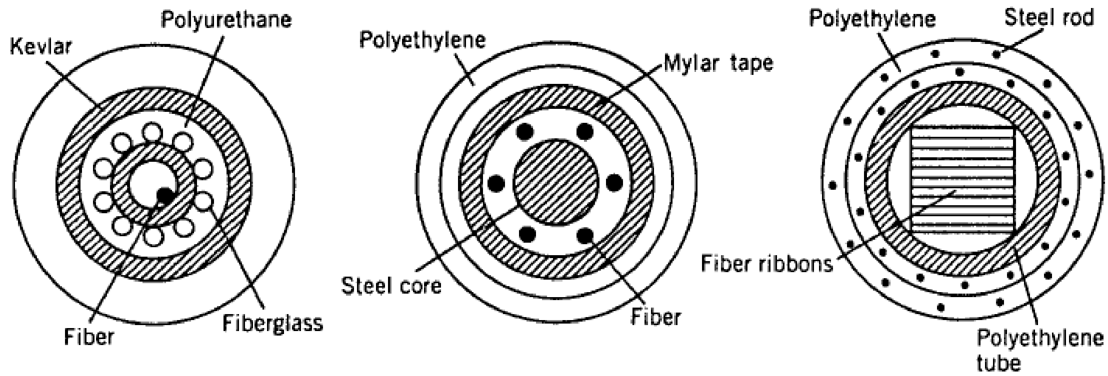

Fig. 1.2 Typical mechanical details of fibre cables.

The basic material of fibre is silica, i.e., SiO_2 which is the most abundant material on earth. But is has to be subjected to purification to obtain no refractive index changes along *z* (propagation axis). Additionally, it has to be doped (mixed with) in a controllable manner to obtain (minute, extremely small) core and cladding refractive index difference. Candidate dopants are B_2O_3, GeO_2, P_2O_5 So possible combinations of core and cladding are

- $GeO_2 + SiO_2$: Core , SiO_2 : Cladding
- $P_2O_5 + SiO_2$: Core , SiO_2 : Cladding
- SiO_2 : Core , $B_2O_3 + SiO_2$: Cladding

During manufacturing, the fibre is usually drawn (and reduced to required core cladding dimensions from a preform rod as shown in Fig 1.3 (directly copied from Fig. 2.22 of Ref [2]).

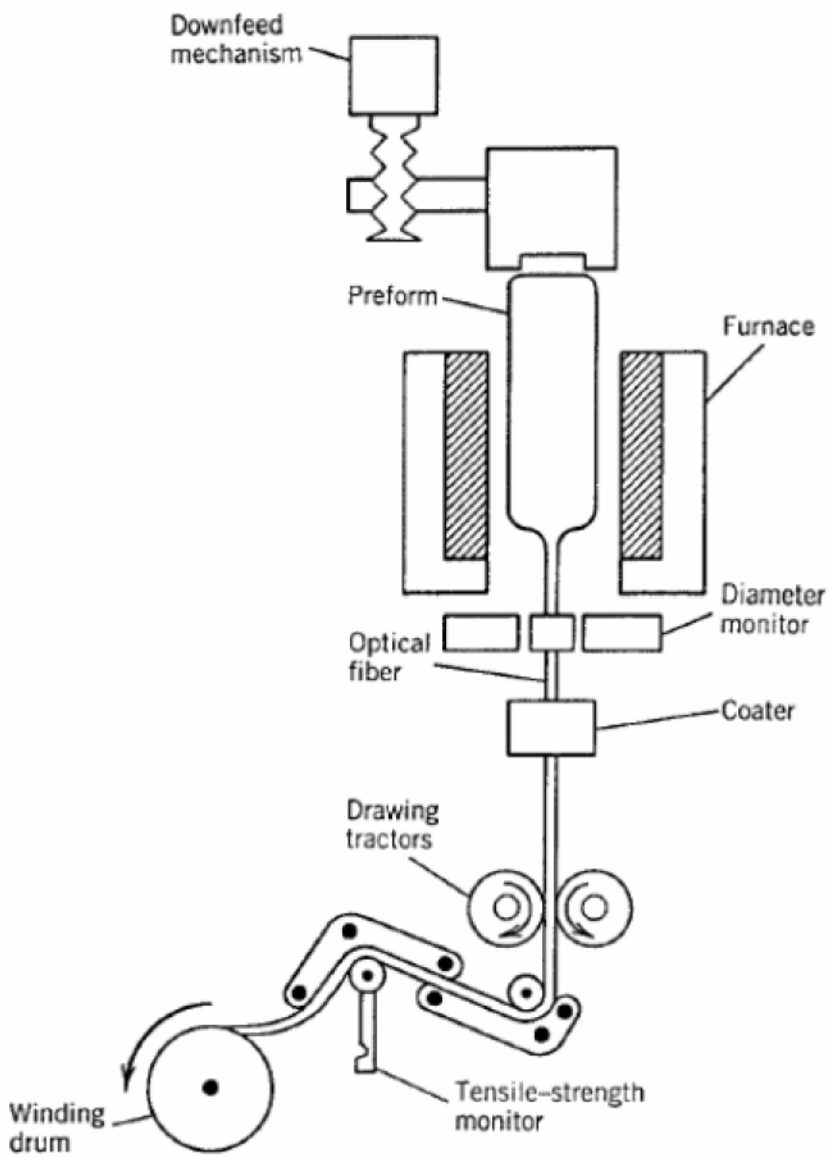

Fig. 1.3 Fibre drawing mechanism.

2. General Description of Attenuation and Dispersion in Fibres

Attenuation and dispersion are two main mechanism in fibres, first (attenuation) being a limitation in maximum achievable link length (without repeaters) the second (dispersion) limiting the maximum achievable bit rate in the fibre link. As expected, both mechanisms are link length related and practically inseparable of course. But from a theoretical analysis point of view, we would like to visualize that attenuation acts along vertical axis, whereas dispersion acts along time axis, as illustrated in Fig. 2.1.

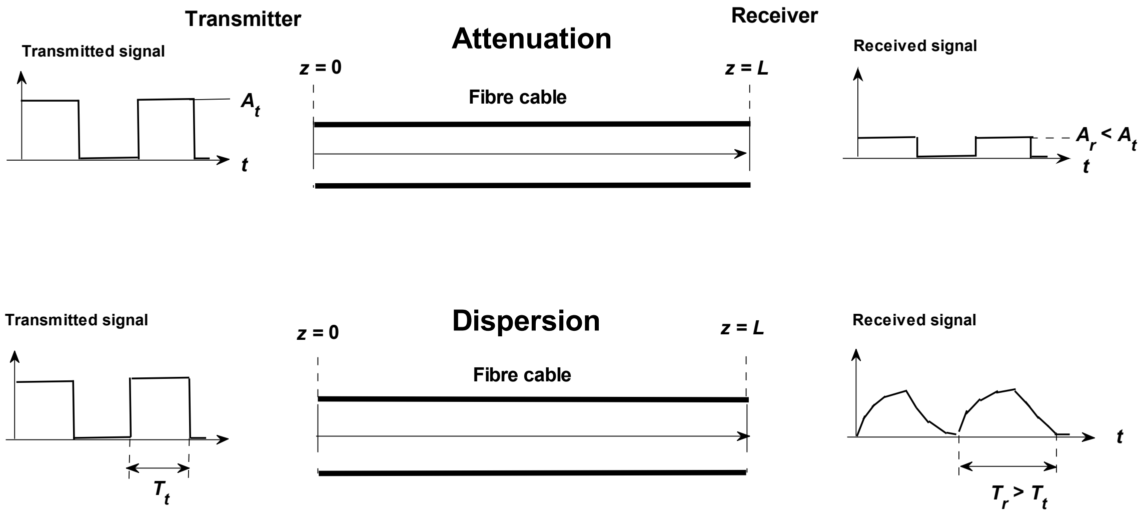

Fig 2.1 Illustration of the effects of attenuation and dispersion on received signal.

3. Attenuation in Fibres

Attenuation in fibres is the result of several mechanisms. For this, we present two graphs as given in Fig 3.1 and 3.2. In Fig. 3.1 the different loss mechanisms are stated and the related curves are given.

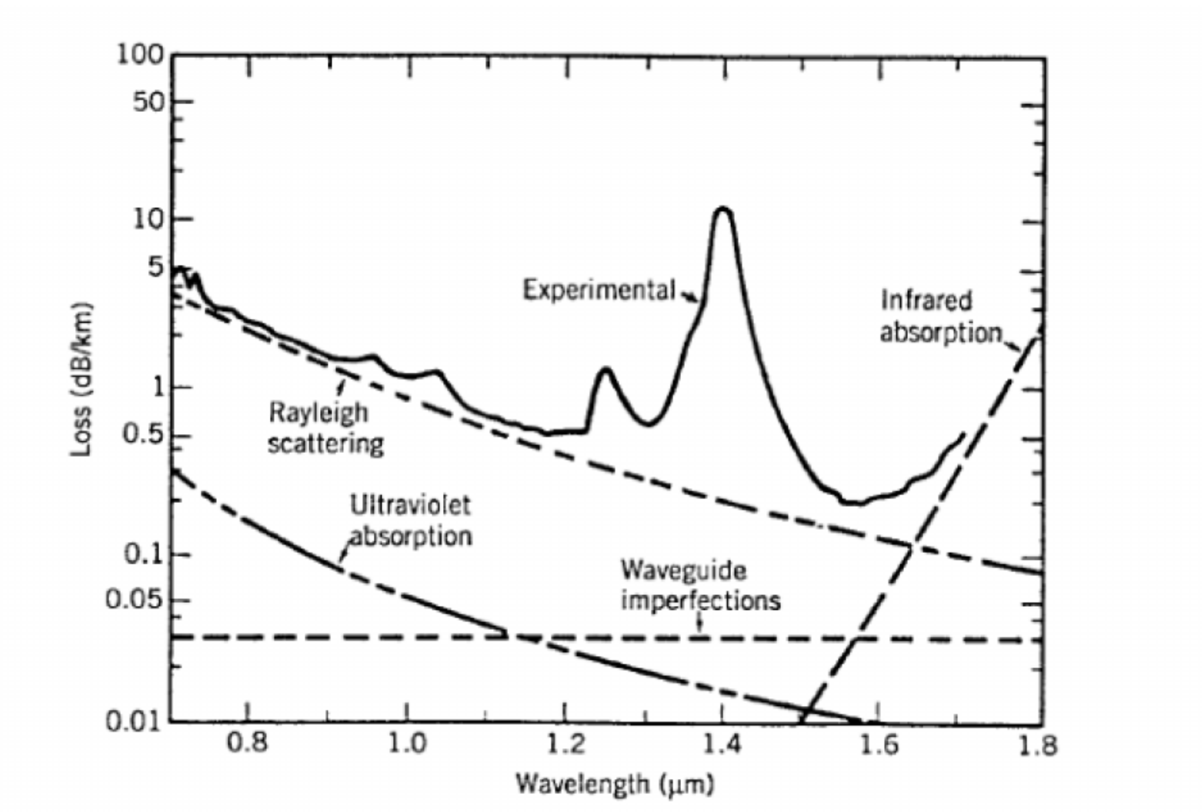

Fig. 3.1 The fibre attenuation graphs (copied directly from Fig. 2.15 of Ref [2], for single mode fibre, 1979).

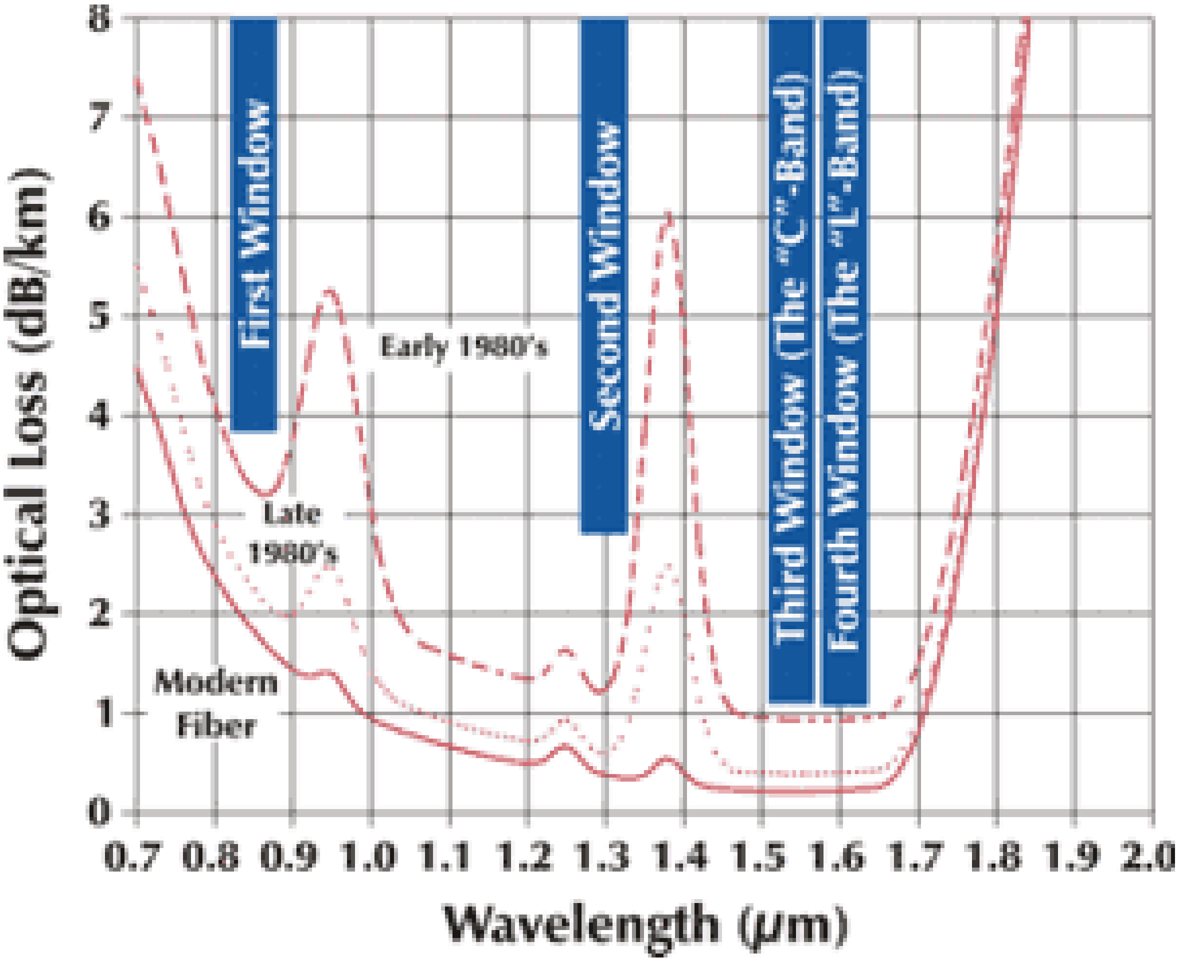

Fig. 3.2 The progress in lowering fibre attenuation throughout the years and low loss windows (copied from http://www.fiber-optics.info/history/P2).

In Fig. 3.2 on the other hand, the progress in lowering fibre attenuation is shown as well as the operational windows. Hence we see that it is reasonable to operate around $\lambda = 1.3$. In fact, single wavelength operations have long favoured $\lambda = 1.55$.

The optical power along fibre axis will decay exponentially. Thus the power at $z = L$, i.e., $P(z = L)$ will be related to the power at $z = 0$, i.e., $P(z = 0)$ as

$$P(z = L) = \exp(-\alpha_p L) P(z = 0) \tag{3.1}$$

Thus the attenuation coefficient, α_p will be

$$\alpha_p = \frac{1}{L} \ln\left[\frac{P(z = 0)}{P(z = L)}\right] \tag{3.2}$$

As seen from (3.2), α_p has then the units of km^{-1}. It is customary to express the attenuation coefficient in dB / km as indicated on the vertical axes of Figs. 3.1 and 3.2. Denoting the attenuation coefficient of dB / km as α , the two attenuation coefficients α_p and α will be related as

$$\alpha\left(\mathrm{dB\ /\ km}\right) = \frac{10}{L}\log_{10}\left[\frac{P(z=0)}{P(z=L)}\right] = \frac{10}{L}\log_{10}\left[\exp\left(\alpha_p L\right)\right]$$
$$\alpha\left(\mathrm{dB\ /\ km}\right) = 4.343\alpha_p\left(\mathrm{km}^{-1}\right) \tag{3.3}$$

Example 3.1 : A fibre has $\alpha = 0.8\ \mathrm{dB\ /\ km}$ and transmits light at $\lambda = 1300\ \mathrm{nm}$, the power on launch, i.e., $P(z=0) = 200$. Find the light power after $z = L = 30\ \mathrm{km}$.

Solution 1 : Using the second line of (3.3), we find

$$\alpha_p = \frac{\alpha}{4.343} \simeq 0.1842 \quad , \quad P(z = L = 30\ \mathrm{km}) = \exp\left(-\alpha_p L\right)P(z=0)$$
$$P(z = L = 30\ \mathrm{km}) = \exp(-0.1842\times 30)\times 200\times 10^{-6} \simeq 0.8 \tag{3.4}$$

Solution 2 : Alternatively using the first line (3.3)

$$\alpha\left(\mathrm{dB\ /\ km}\right)L = 10\log_{10}\left[\frac{P(z=0)}{P(z=L)}\right] = 10\log_{10}\left[P(z=0)\right] - 10\log_{10}\left[P(z=L)\right]$$
$$\alpha\left(\mathrm{dB\ /\ km}\right)L = P(z=0)(\text{in dBm}) - P(z=L)(\text{in dBm})$$
$$P(z=L)(\text{in dBm}) = P(z=0)(\text{in dBm}) - \alpha\left(\mathrm{dB\ /\ km}\right)L$$
$$= -6.9897\ \mathrm{dBm} - 24\ \mathrm{dB} = -30.9897\ \mathrm{dBm}$$
$$P(L = 30\ \mathrm{km}) = 10^{-3.09897} = 0.79 \tag{3.5}$$

As seen the results of (3.4) and (3.5) agree quite well.

Exercise 3.1 : A fibre link has a length of $L = 40\ \mathrm{km}$. The transmitted optical power is $P(z=0) = 1\ \mathrm{mW}$, while the received optical power is $P(L = 40\ \mathrm{km}) = 0.5$. Find α_p and $\alpha\left(\mathrm{dB\ /\ km}\right)$ of this fibre.

4. Dispersion in Fibres

Attenuation acts along the amplitude axis, causing the signal power (amplitude) to fall as the light propagates along fibre axis.

Dispersion on the other acts along time axis causing pulse broadening as illustrated on the second line of Fig. 2.1. The originates from the presence of many modes propagating simultaneously in the fibre. These modes will possess different propagation constants, thus will arrive at different times at the receiving side of the fibre link. This means that there are time delay differences between modes. Since each mode carries the electrical message signal independently, different arrival times or delay differences will cause a broadening as illustrated on the second line of Fig. 2.1. Four types of dispersions exist in fibres. These are

a) Intermodal dispersion : Present due the existence of many different propagating modes in the fibre. So this effect is applicable to multimode step index and graded index fibres and it is absent in single mode fibres.
b) Intramodal dispersion, Chromatic dispersion : As the name implies, this is basically due to the presence of many wavelengths (chroma). A light source will practically radiate light in a finite spectrum, rather than a single wavelength. Thus the frequency spectrum of a light source will not be a delta function but will extend over a range of frequencies (say *N* wavelengths such as $\lambda_1,\ \lambda_2 \cdots \lambda_N$). On the other hand, the refractive index is known to variy with wavelength. As a results we obtain two different consequences ;
    1. Material dispersion : The dependence of the refractive index on wavelength converts the fibre many fibres each having a refractive index of $n(\lambda_1),\ n(\lambda_2)\cdots n(\lambda_N)$.
    2. Waveguide dispersion : A source of many wavelengths can be represented as individual sources each having radiation wavelengths of $\lambda_1,\ \lambda_2 \cdots \lambda_N$. This way for a single fibre, there appear to be many light sources.
c) Polarization mode dispersion: This type of dispersion arises due time delays between the two polarized versions of the same mode.

The pictorial representation of the intramodal (choromatic) dispersion is shown in Fig. 4.1.

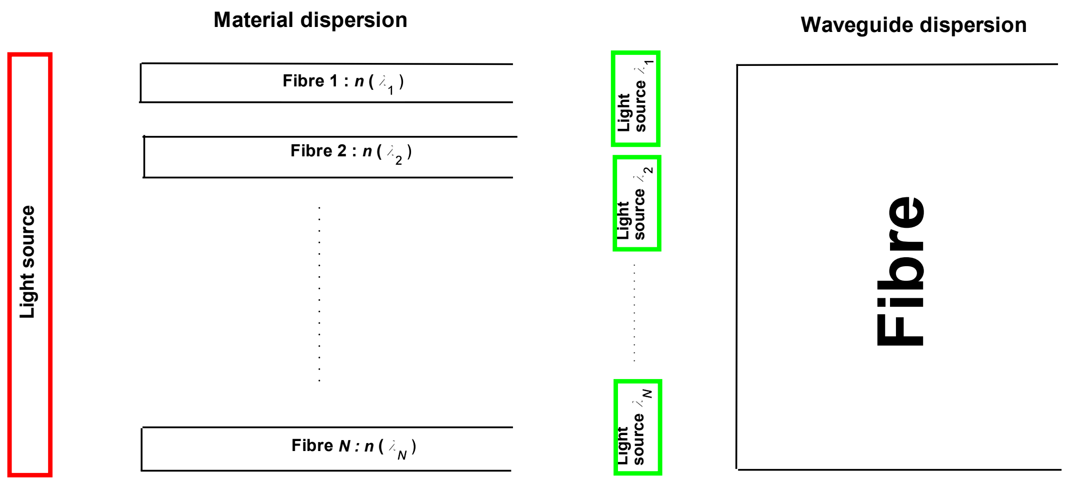

Fig. 4.1 Pictorial representation of material and waveguide dispersions.

It is important to realize that all types of dispersions will exist in multimode fibres, but single mode fibres will not have intermodal dispersion, since theoretically they support one mode only. For ease of apprehension, we start with intermodal dispersion.

4.1 Intermodal Dispersion

To estimate the maximum intermodal dispersion that can occur, we take the simple case of finding the delay between the fastest and the slowest ray in a step index (multimode) fibre. This situation is depicted in Fig. 4.1.1.

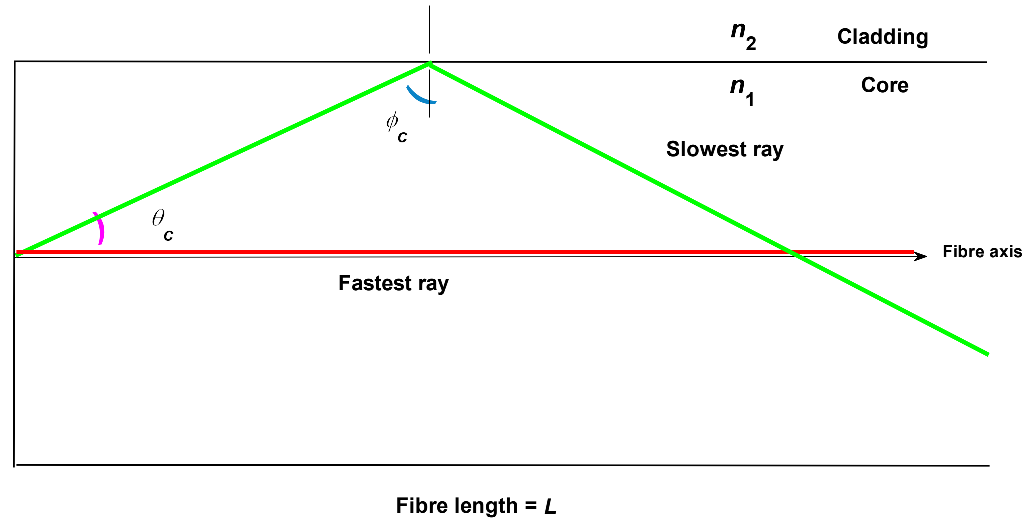

Fig 4.1.1 Illustration of fastest and slowest rays in a step index multimode fibre.

According to Fig. 4.1.1, the fastest ray will be the ray propagating parallel to fibre axis or along the fibre axis, so the time taken for this fastest ray to reach the receiver on other side of the fibre is

$$T_{fast} = \frac{\text{Fibre path length}}{\text{Speed of ray in fibre}} = \frac{L}{c/n_1} = \frac{Ln_1}{c} \qquad (4.1.1)$$

The slowest ray will be the ray at the critical angle (i.e. the ray that has the critical angle for total internal reflection). From (2.2) of Notes on Fibre Propagation_Jan 2013_HTE, the cosine of this angle is related to the refractive index of the core an cladding as follows

$$\cos(\theta_c) = \frac{n_2}{n_1} \qquad (4.1.2)$$

It is easy to see that, in the case of slowest ray, fibre path length will be $L/\cos(\theta_c)$, then the time taken fort his ray to traverse the fibre length and reach the receiver is

$$T_{slow} = \frac{L/\cos(\theta_c)}{c/n_1} = \frac{Ln_1}{c\cos(\theta_c)} \qquad (4.1.3)$$

Now, by defining the time delay between T_{slow} and T_{fast} as

$$\delta T = T_{slow} - T_{fast} \qquad (4.1.4)$$

Using (4.1.1) to (4.1.3), ΔT in (4.1.4) will become

$$\delta T = T_{slow} - T_{fast} = \frac{Ln_1}{c}\left(\frac{n_1}{n_2} - 1\right) \simeq \frac{Ln_1}{cn_2}\Delta n_1 \simeq \frac{L\mathrm{NA}^2}{2cn_2} \quad (4.1.5)$$

where we have used the approximation and the equivalence of

$$n_1 - n_2 \simeq \Delta n_1 \quad , \quad \mathrm{NA}^2 = 2\Delta n_1 \quad (4.1.6)$$

As seen from (4.1.5), the time delay is directly proportional to fibre length L. So δT indicates the total dispersion in the fibre. Scaling δT by the fibre length L, we obtain the intermodal dispersion of the fibre per km

$$\frac{\delta T}{L} = \frac{\mathrm{NA}^2}{2cn_2} \quad \text{in ns / km} \quad (4.1.7)$$

The inverse of δT and $\frac{\delta T}{L}$ can associated with the total bandwidth, B_t of and the bandwidth offered over 1 km of the fibre, B_k. Related to these quantities, we can also define the corresponding fibre capacity C_t and C_k as follows

$$\begin{aligned} &B_t = \frac{1}{\delta T} \ , \ B_k = \frac{L}{\delta T} \ , \ B_t = \frac{B_k}{L} \quad B_t \text{ in MHz} , \quad B_k \text{ in MHz} \times \text{km} , L \text{ in km} \\ &C_t \simeq B_t = \frac{1}{\delta T} = \frac{2cn_2}{L\mathrm{NA}^2} \quad , \quad C_k \simeq B_k = \frac{L}{\delta T} = \frac{2cn_2}{\mathrm{NA}^2} \quad C_t \text{ in Mb / s} \quad C_k \text{ in Mb/s} \times \text{km} \\ &C_t = \frac{C_k}{L} \quad \text{or} \quad C_k = C_t L \end{aligned} \quad (4.1.8)$$

Example 4.1.1 : A fibre has $n_1 = 1.485$ and $n_2 = 1.47$. If this fibre is used at a length of $L = 30$ km, find the total intermodal dispersion, intermodal dispersion per km and the bit rate capacity that fibre offers.

Solution : From given n_1 and n_2, we initially calculate the numerical aperture, NA

$$\mathrm{NA} = \left(n_1^2 - n_2^1\right)^{0.5} = \left(1.485^2 - 1.47^2\right)^{0.5} = 0.2105 \quad (4.1.9)$$

Using (4.1.5), (4.1.7) and (4.1.8)

$$\delta T = \frac{L\text{NA}^2}{2cn_2} = 1.5077\times10^{-6}\ \text{s} = 1.5077$$

$$\frac{\delta T}{L} = \frac{\text{NA}^2}{2cn_2} = 50.255\ \text{ns / km}$$

$$B_t = \frac{1}{\delta T} = 0.66326\ \text{MHz : Total bandwidth offered over 30 km}$$

$$B_k = \frac{L}{\delta T} = 19.898\ \text{MHz}\times\text{km : Bandwidth offered over 1 km}$$

$$C_t \simeq B_t = 0.66326\ \text{Mb/s : Total capacity offered over 30 km}$$

$$C_k \simeq B_k = 19.898\ \text{Mb/s}\times\text{km : Capacity offered over 1 km} \qquad (4.1.10)$$

Example 4.1.2 : Assume that fibre in Example 4.1.1, is to be operated at full length of $L = 30\ \text{km}$, find the amount of source power necessary to achieve an SNR of 30 dB at receiver, if $\alpha = 0.8\ \text{dB / km}$.

Solution : It is much easier in this case to work backwards from the receiver side to the transmitter. Assuming we work at room temperature of $20\ ^0\text{C}$, then

$$T_a\left(\text{absolute temprature}\right) = 273 + 20 = 293\ ^0\text{K}$$

$$S_n\left(f\right) = \frac{kT_a}{2} = \frac{1.38\times10^{-23}\times293}{2} = 2.0217\times10^{-21}\ \text{J : Two side noise spectral density}$$

$$P_n = 2S_n\left(f\right)B_t = 2\times2.0217\times10^{-21}\times0.66326\times10^6 = 2.6818\times10^{-15}\ \text{W : Total noise power}$$

$$P\left(z=L\right) = P_n\times\text{SNR} = 2.6818\times10^{-15}\times10^3 = 2.6818\times10^{-12}\ \text{W: Minimum received power for SNR} = 30\ \text{dB}$$

$$P\left(z=L\right)\left(\text{in dBm}\right) = -85.7157\ \text{dBm}$$

$$P\left(z=0\right)\left(\text{in dBm}\right) = P\left(z=L\right)\left(\text{in dBm}\right) + \alpha L = -61.7157\ \text{dBm}$$

$$P\left(z=0\right) = 10^{-6.17157} = 0.665\ \mu\text{W : Minimum power to be launched into the fibre at transmitter} \qquad (4.1.11)$$

In practice, there will be additional losses such as light source to fibre coupling loses, bending losses so on. This way the light source needs to deliver somewhat higher than calculated on the last line of (4.1.11)

Exercise 4.1.2 : Repeat the calculations in Example 4.1.2 if the fibre length is extended to 40 km. Note that in such a case, the bandwidth calculation in Example 4.1.1 has to be revised as well.

To study intramodal dispersion, we have to introduce the term group velocity.

4.2 Derivation of Group Velocity

Initially we take a monochormatic plane wave at the point of $\mathbf{R} = 0$ in Cartesian coordinates of $\mathbf{R} = \left(x,\ y,\ z\right)$

$$U\left(\mathbf{R}=0,t\right) = A\exp\left(j2\pi f_1 t\right) \qquad (4.2.1)$$

(4.2.1) is indeed a a monochormatic plane wave since, it contains a single frequency of oscillations, f_1 and it has no amplitude variations against time or the spatial coordinates, i.e., A is independent of time and $\mathbf{R} = (x,\, y,\, z)$. Assume that the plane wave of (4.2.1) travels a distance of $\mathbf{R}$ in free space (where $n = 1$) and but if paraxial approximation is valid, i.e., wave propagates almost confined around z axis, so $\mathbf{R} \simeq z$. Due to this travel, a time of z/c passes, then at this new location of $\mathbf{R}$ the plane wave can be written as

$$
\begin{aligned}
U(\mathbf{R} \simeq z, t) &= A \exp\left[j2\pi f_1 \left(t - \frac{z}{c} \right) \right] \\
&= A \exp(j2\pi f_1 t) \exp\left(-j2\pi f_1 \frac{z}{c} \right) \\
&= A \exp(j2\pi f_1 t) \exp(-jk_1 z)
\end{aligned}
$$

$$
k_1 = \frac{2\pi f_1}{c} = \frac{\omega_1}{c} = \frac{2\pi}{\lambda_1} \tag{4.2.2}
$$

This way time lapse is embedded into k_1. Next we consider a pulsed waveform of

$$
U(\mathbf{R} = 0, t) = A(t) \exp(j2\pi f_1 t) \tag{4.2.3}
$$

This waveform is longer monochromatic, but will occupy a frequency band around f_1, depending on the form of $A(t)$. Alternatively (4.2.3) will represent a polychromatic source with a central frequency f_1. If we wish to find $U(\mathbf{R} \simeq z, t)$ corresponding to (4.2.3), similar to the first line of (4.2.2), we just replace t in (4.2.3) by $t - z/c$. Hence for the polychromatic source of (4.2.3), $U(\mathbf{R} \simeq z, t)$ will become

$$
\begin{aligned}
U(\mathbf{R} \simeq z, t) &= A\left(t - \frac{z}{c} \right) \exp\left[j2\pi f_1 \left(t - \frac{z}{c} \right) \right] \\
&= A\left(t - \frac{z}{c} \right) \exp(j2\pi f_1 t) \exp(-jk_1 z)
\end{aligned} \tag{4.2.4}
$$

(4.2.4) is valid for propagation in free space where the refractive index is 1. When a polychromatic light is launched into a medium, with refractive index n, the following replacements should be made

$$
c \rightarrow c_n = \frac{c}{n} \quad , \quad k_1 \rightarrow k_n = \frac{2\pi f_1 n}{c} = \frac{\omega_1 n}{c} \rightarrow \beta \tag{4.2.5}
$$

In this case, the field at in (4.2.4) at $\mathbf{R} \simeq z$

$$
U(\mathbf{R} \simeq z, t) = A\left(t - \frac{zn}{c} \right) \exp(j2\pi f_1 t) \exp(-j\beta z) \tag{4.2.6}
$$

(4.2.6) indicates that there is a more time delay in a medium of $n>1$ (which is the case in practice). Note that in propagation mediums other than free space, it is customary to denote the propagation constant as β instead of k (also known as wave number).

Now represent the polychromatic source as the sum of two complex exponentials at frequencies of f_1 and f_2, having a time constant amplitude function

$$U(\mathbf{R}=0,t)=A\left[\exp(j2\pi f_1 t)+\exp(j2\pi f_2 t)\right] \qquad (4.2.7)$$

(4.2.7) is in the form of amplitude modulation (AM) and can be written as such by taking the real part of the exponentials

$$\begin{aligned} U_r(\mathbf{R}=0,t) &= A\left[\cos(2\pi f_1 t)+\cos(2\pi f_2 t)\right] \\ &= 2A\cos\left[2\pi\left(\frac{f_1-f_2}{2}\right)t\right]\cos\left[2\pi\left(\frac{f_1+f_2}{2}\right)t\right] \end{aligned} \qquad (4.2.8)$$

(4.2.8) constitutes DSB (a type of AM) with the envelope of $\cos\left[2\pi\left(\frac{f_1-f_2}{2}\right)t\right]$. In a similar manner to the above development, after propagating a distance of z (4.2.8) becomes

$$\begin{aligned} &U(\mathbf{R}\simeq z,t)=A\left[\exp(j2\pi f_1 t-jk_1 z)+\exp(j2\pi f_2 t-jk_2 z)\right] \\ &k_1=\frac{2\pi f_1 n}{c}=\frac{\omega_1 n}{c} \quad , \quad k_2=\frac{2\pi f_2 n}{c}=\frac{\omega_2 n}{c} \end{aligned} \qquad (4.2.9)$$

Again when only the real part is considered, (4.2.9) will become

$$U(\mathbf{R}\simeq z,t)=2A\cos\left[2\pi\left(\frac{f_1-f_2}{2}\right)t-(k_1-k_2)z\right]\cos\left[2\pi\left(\frac{f_1+f_2}{2}\right)t-(k_1+k_2)z\right] \qquad (4.2.10)$$

The first term in (4.2.10) will again correspond to the envelope. In analogy with the definition of the propagation constant on the last line of (4.2.2), we can define equivalent k and equivalent velocity for (4.2.10) as

$$\begin{aligned} &k_1=\frac{2\pi f_1}{c}=\frac{\omega_1}{c} \quad , \quad k_{eq}=\frac{2\pi(f_1-f_2)}{c_{eq}}=k_1-k_2 \\ &c_{eq}\rightarrow V_g=\frac{2\pi(f_1-f_2)}{k_1-k_2}=\frac{\omega_1-\omega_2}{k_1-k_2} \end{aligned} \qquad (4.2.11)$$

where V_g is popularly known as group velocity. Now if we are in a dispersive media, which means the refractive index is wavelength dependent, then we can talk about the following dependencies

$$\begin{aligned} &n\rightarrow n(\lambda) \quad , \quad n(\lambda) \\ &\lambda(n) \quad , \quad f(n) \quad , \quad \omega(n) \quad , \quad \omega(k) \end{aligned} \qquad (4.2.12)$$

In the light of the dependencies in (4.2.12), it is possible to write V_g of (4.2.11) as

$$V_g = \frac{\omega(k_1) - \omega(k_2)}{k_1 - k_2} \tag{4.2.13}$$

By taking a central k such as

$$k = 0.5(k_1 + k_2) \quad , \quad k_1 = 2k - k_2 \quad , \quad k_2 = 2k - k_1 \tag{4.2.14}$$

Using (4.2.14) in (4.2.13) we get

$$V_g = \frac{\omega(2k - k_2) - \omega(2k - k_1)}{k_1 - k_2} \tag{4.2.15}$$

Expanding the terms in the numerator of the right hand side of (4.2.15) around k in Taylor series will give

$$\begin{aligned} V_g &= \frac{\omega(2k - k_2) - \omega(2k - k_1)}{k_1 - k_2} \\ &= \frac{\omega(2k) - k_2\omega'(2k)\cdots - \omega(2k) + k_1\omega'(2k)\cdots}{k_1 - k_2} \\ &\simeq \omega'(2k) \rightarrow \frac{d\omega}{dk} = V_g \end{aligned} \tag{4.2.16}$$

For application in a fibre, we replace k with propagation constant β, thus the group velocity V_g will become

$$V_g = \frac{d\omega}{d\beta} = \left(\frac{d\beta}{d\omega}\right)^{-1} \quad , \quad \frac{1}{V_g} = \frac{d\beta}{d\omega} \tag{4.2.17}$$

4.3 Material and Waveguide Dispersion in Single Mode Fibres

To calculate the material and waveguide dispersion in single mode fibres using (4.2.17), we take the normalized definitions of the propagation constant and its association with the normalized frequency of fibre. They are taken from (3.30) and (3.35) of Notes on Fibre Propagation_Jan 2013_HTE and restated below

$$b_n = \frac{\beta / k - n_2}{n_1 - n_2} = \frac{\bar{n} - n_2}{n_1 - n_2} \quad , \quad b_n(V) \simeq (1.1428 - 0.996/V)^2 \tag{4.3.1}$$

From (4.3.1), we that a new term called mode index, $\bar{n}$ is defined and set to β / k. Thus we can obtain the derivative of the propagation constant with respect to ω as follows

$$\beta = \overline{n}k = \overline{n}\frac{\omega}{c} \quad , \quad \frac{d\beta}{d\omega} = \frac{\overline{n}}{c} + \frac{\omega}{c}\frac{d\overline{n}}{d\omega}$$

$$V_g = \left(\frac{d\beta}{d\omega}\right)^{-1} = \frac{c}{\overline{n} + \omega\dfrac{d\overline{n}}{d\omega}} \tag{4.3.2}$$

Now for a fibre length of L, the time taken to arrive at receiver is $T = L/V_g$. The broadening that will be experienced will be δT. To estimate dispersion created by a source of spectral width σ_ω, we write as follows

$$\delta T = \frac{dT}{d\omega}\sigma_\omega = \frac{d}{d\omega}\left(\frac{L}{V_g}\right)\sigma_\omega = L\frac{d^2\beta}{d\omega^2}\sigma_\omega = L\beta_2\sigma_\omega \quad , \quad \beta_2 = \frac{d^2\beta}{d\omega^2} \tag{4.3.3}$$

$\beta_2 = \dfrac{d^2\beta}{d\omega^2}$ is known as group velocity dispersion (GVD) of the fibre. We can also formulate (4.3.3) in terms of λ, instead of ω. For this, we use the following conversions

$$\omega = 2\pi f \;\; , \; d\omega = 2\pi df \;\; , \; \frac{d}{df} = 2\pi\frac{d}{d\omega}$$

$$\omega = 2\pi\frac{c}{\lambda} \;\; , \; d\omega = -\frac{2\pi c}{\lambda^2}d\lambda \;\; , \; \frac{d}{d\lambda} = -\frac{2\pi c}{\lambda^2}\frac{d}{d\omega}$$

$$\sigma_\omega \to 2\pi\sigma_f \to -\frac{2\pi c}{\lambda^2}\sigma_\lambda$$

$$\delta T = \frac{d}{d\omega}\left(\frac{L}{V_g}\right)\sigma_\omega = \frac{d}{d\lambda}\left(\frac{L}{V_g}\right)\sigma_\lambda = -\frac{2\pi c}{\lambda^2}\beta_2\sigma_\lambda = DL\sigma_\lambda$$

$$D = -\frac{2\pi c}{\lambda^2}\frac{d}{d\omega}\left(\frac{1}{V_g}\right) = \frac{d}{d\lambda}\left(\frac{1}{V_g}\right) = -\frac{2\pi c}{\lambda^2}\beta_2 \tag{4.3.4}$$

D is called the dispersion parameter on which we concentrate from this point onwards. Note that the definition of β_2 in (4.3.4) is still the same as in (4.3.3), i.e. $\beta_2 = \dfrac{d^2\beta}{d\omega^2}$. Substituting for β in β_2 from (4.3.2), the dispersion parameter D in (4.3.4) becomes

$$D = -\frac{2\pi c}{\lambda^2}\frac{d}{d\omega}\left(\frac{1}{V_g}\right) = -\frac{2\pi c}{\lambda^2}\frac{d^2\beta}{d\omega^2} = -\frac{2\pi c}{\lambda^2}\frac{d}{d\omega}\left(\frac{\overline{n}}{c} + \frac{\omega}{c}\frac{d\overline{n}}{d\omega}\right)$$

$$= -\frac{2\pi}{\lambda^2}\left(2\frac{d\overline{n}}{d\omega} + \omega\frac{d^2\overline{n}}{d\omega^2}\right) = D_M + D_W \tag{4.3.5}$$

where on the last line, we have split up the dispersion parameter D into two parts, namely material dispersion is denoted by D_M, while waveguide dispersion part is denoted by D_W. But note that these are not exactly the two terms of the left hand side of the last line of (4.3.5), rather a mixture, as shall be seen later. Now we use the definition of $\overline{n}$ in terms of b_n and obtain the following

$$b_n = \frac{\overline{n} - n_2}{n_1 - n_2} \quad , \quad n_1 - n_2 \simeq \Delta n_1 \simeq \Delta n_2 \quad , \quad \overline{n} \simeq n_2 + \Delta n_2 b_n = \overline{n}_2 + \overline{n}_b \tag{4.3.6}$$

In the rightmost expression n_2 is the cladding index and its variation with the wavelength will result in material dispersion described in 4b1), while the second term involves normalized frequency related parameter b_n will hence the dispersion variation between the different HE_{11} modes of the fibre, this term will then be responsible for waveguide dispersion. This way

$$D_M \to 2\frac{d\overline{n}_2}{d\omega} + \omega\frac{d^2\overline{n}_2}{d\omega^2} = 2\frac{dn_2}{d\omega} + \omega\frac{d^2 n_2}{d\omega^2} = \frac{d}{d\omega}\overbrace{\left(n_2 + \omega\frac{dn_2}{d\omega}\right)}^{n_{2g}} = \frac{d}{d\omega}n_{2g} \tag{4.3.7}$$

Using (4.3.5) and (4.3.7), D_M will be

$$D_M = -\frac{2\pi}{\lambda^2}\frac{d}{d\omega}\left(n_2 + \omega\frac{dn_2}{d\omega}\right) = -\frac{2\pi}{\lambda^2}\frac{d}{d\omega}n_{2g} = \frac{1}{c}\frac{d}{d\lambda}n_{2g} = -\frac{\lambda}{c}\frac{d^2 n_2}{d\lambda^2} \tag{4.3.8}$$

From (4.3.8), we conclude that material dispersion is simply related to the second derivative of the cladding refractive index with respect to wavelength of the source. For the waveguide dispersion on the other hand, from (4.3.6) we take $\overline{n}_b$ part, then

$$\begin{aligned} D_W &= -\frac{2\pi}{\lambda^2}\left(2\frac{d\overline{n}_b}{d\omega} + \omega\frac{d^2\overline{n}_b}{d\omega^2}\right) = -\frac{2\pi}{\lambda^2}\left[2\frac{d\left(\Delta n_2 b_n\right)}{d\omega} + \omega\frac{d^2\left(\Delta n_2 b_n\right)}{d\omega^2}\right] \\ &= -\frac{2\pi\Delta}{\lambda^2}\left[2b_n\frac{dn_2}{d\omega} + 2n_2\frac{db_n}{d\omega} + \omega\frac{d}{d\omega}\left(b_n\frac{dn_2}{d\omega} + n_2\frac{db_n}{d\omega}\right)\right] \end{aligned} \tag{4.3.9}$$

$\frac{db_n}{d\omega}$ in (4.3.9) has to be converted into $\frac{db_n}{dV}$, since as seen from (4.3.1) b_n is expressed in terms of V. To do this we utilize the followings

$$\begin{aligned} V &= kan_1\sqrt{2\Delta} = \frac{\omega}{c}an_1\sqrt{2\Delta} \\ \frac{d}{d\omega} &= \frac{1}{c}an_1\sqrt{2\Delta}\frac{d}{dV} = \frac{k}{\omega}an_1\sqrt{2\Delta}\frac{d}{dV} = \frac{V}{\omega}\frac{d}{dV} \end{aligned} \tag{4.3.10}$$

With the use of (4.3.10) in (4.3.9), we get the waveguide dispersion as

$$D_W = -\frac{2\pi\Delta}{\lambda^2}\left[\frac{n_{2g}^2}{n_2\omega}V\frac{d^2\left(Vb_n\right)}{dV^2} + \frac{dn_{2g}}{d\omega}\frac{d\left(Vb_n\right)}{dV}\right] \tag{4.3.11}$$

So to calculate waveguide dispersion we need the graphs of n_{2g}, b_n, $\frac{d\left(Vb_n\right)}{dV}$ and $\frac{d^2\left(Vb_n\right)}{dV^2}$. The last two can be worked out from $b_n\left(V\right)$, but graphs are easier to use. They are provided below (copied directly from Figs. 2.8 and 2.9 of Ref. [2]).

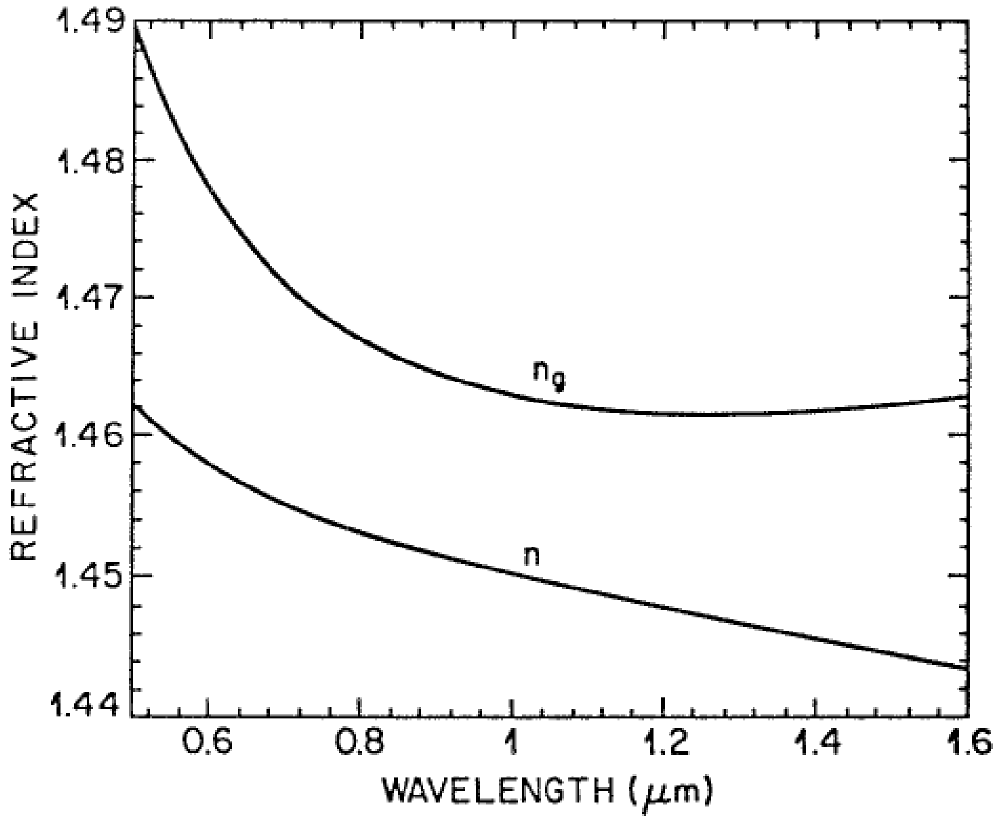

Fig. 4.3.1 The graphs of n (can be used for n_2) and n_{2g} against wavelength.

So after calculating D_M form (4.3.8) and D_W from (4.3.11), finally we find the total intermodal dispersion δT using (4.3.4) and (4.3.5). The results are summarized below in (4.3.12).

$$\begin{aligned}
&\text{Total intermodal dispersion}: \delta T = DL\sigma_\lambda = \left(D_M + D_W\right)L\sigma_\lambda \\
&\text{Material dispersion parameter}: D_M = \frac{1}{c}\frac{d}{d\lambda}n_{2g} = -\frac{\lambda}{c}\frac{d^2 n_2}{d\lambda^2} \\
&\text{Waveguide dispersion parameter}: D_W = -\frac{2\pi\Delta}{\lambda^2}\left[\frac{n_{2g}^2}{n_2\omega}V\frac{d^2\left(Vb_n\right)}{dV^2} + \frac{dn_{2g}}{d\omega}\frac{d\left(Vb_n\right)}{dV}\right] \qquad (4.3.12)
\end{aligned}$$

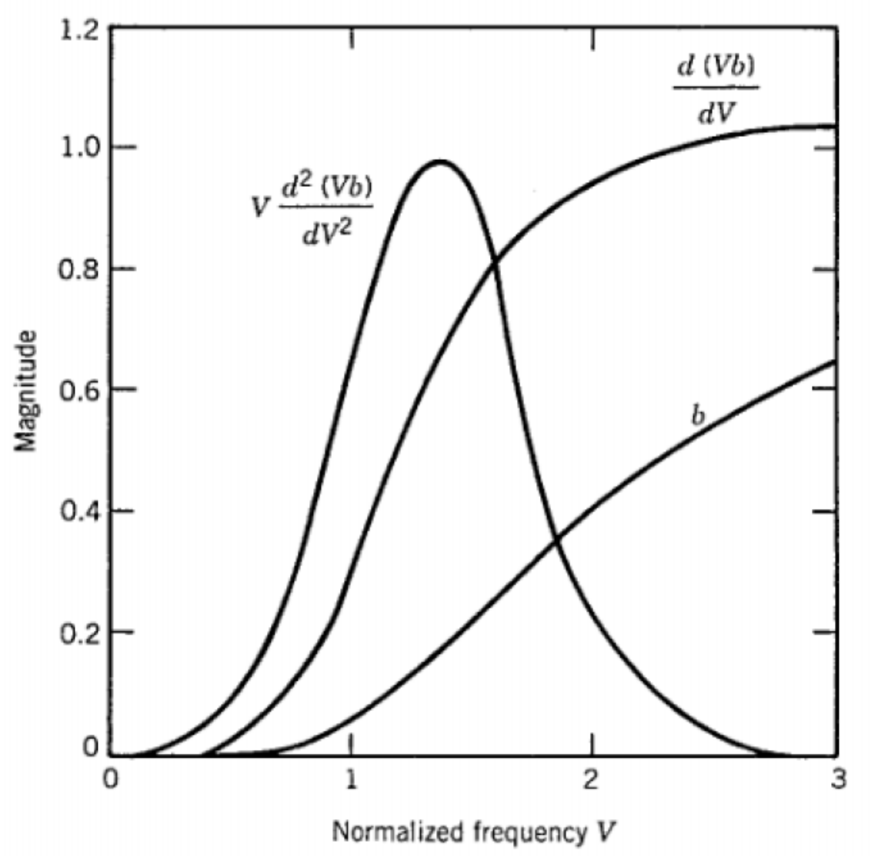

Fig. 4.3.2 The graphs of b_n, $\frac{d\left(Vb_n\right)}{dV}$ and $\frac{d^2\left(Vb_n\right)}{dV^2}$. Note that in our notation b is b_n.

Example 4.3.1 : For a sample calculation of dispersion, we resort to the solution EEM474MT-05.04.2010_CC.pdf, also available on course webpage.

These notes are based on

1) Gerd Keiser, “Optical Fiber Communications” 3nd Ed. 2000, McGraw Hill, ISBN : 0-07-116468-5.
2) Govind P. Agrawal, “Fiber-Optic Communication Systems” 2002, Jon Wiley and Sons, ISBN : 0-471-21571-6.
3) B. E. A. Saleh, M.C. Teich “Fundamentals of Photonics” , 2007, Wiley, ISBN No : 978-0-471-35832-9.
4) My own ECE 474 Lecture Notes.